Anomalous Kinetics in Velocity Space: equations and models

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Equation for anomalous diffusion in momentum space, recently obtained in [1], is solved for the stationary and non-stationary cases on basis of the appropriate probability transition function (PTF). Consideration of diffusion for heavy particles in a gas of the light particles can be essentially simplified due to small ratio of the masses of the particles. General equation for the distribution of the light particles, shifted in velocity space, is also derived. For the case of anomalous diffusion in momentum space the closed equation is formulated for the Fourier-component of the momentum distribution function. The effective friction and diffusion coefficients are found also for the shifted distribution. If the appropriate integrals are finite the equations derived in the paper are applicable for both cases: the PT-function with the long tails and the short range PT-functions in momentum space. In the last case the results are equivalent to the Fokker-Planck equation. Practically the new results of this paper are applicable to strongly non-equilibrium physical systems.

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I. INTRODUCTION

Interest in anomalous diffusion is conditioned by a large variety of applications: semiconductors, polymers, some granular systems, plasmas in specific conditions, various objects in biological systems, physical-chemical systems, et cetera.

Many concrete problems of anomalous diffusion in coordinate space have been solved on the basis of equations with the fractional derivatives. Recently in [2] the new approach to

anomalous diffusion in coordinate space has been formulated. This approach simplifies the problem and at the same time permits to consider the more complicated kernels (PTF in coordinate space). In the present paper, as well as in [1], we apply the similar approach to solve the problem in momentum space.

The deviation from the linear in time $< r^2(t) >\sim t$ dependence of the mean square displacement have been experimentally observed, in particular, under essentially non-equilibrium conditions or for some disordered systems. The average square separation of a pair of particles passively moving in a turbulent flow grows, according to Richardson's law, with the third power of time [3]. For diffusion typical for glasses and related complex systems [4] the observed time dependence is slower than linear. These two types of anomalous diffusion obviously are characterized as superdiffusion $< r^2(t) >\sim t^\alpha$ $(\alpha > 1)$ and subdiffusion $(\alpha < 1)$ [5]. For a description of these two diffusion regimes a number of effective models and methods have been suggested. The continuous time random walk (CTRW) model of Scher and Montroll [6], leading to strongly subdiffusion behavior, provides a basis for understanding photoconductivity in strongly disordered and glassy semiconductors. The Levy-flight model [7], leading to superdiffusion, describes various phenomena as self-diffusion in micelle systems [8], reaction and transport in polymer systems [9] and is applicable even to the stochastic description of financial market indices [10]. For both cases the so-called fractional differential equations in coordinate and time spaces are applied as an effective approach [11].

However, recently a more general approach has been suggested in [2], [12], which avoid the fractional differentiation, reproduce the results of the standard fractional differentiation method, when the last one is applicable, and permit to describe the more complicated cases of anomalous diffusion processes. In [13] these approach has been applied also to the diffusion in the time-dependent external field.

In this paper the problem of anomalous diffusion in the momentum (velocity) space will be considered. In spite of formal similarity, diffusion in the momentum space is very different physically from the coordinate space diffusion. It is clear already because the momentum conservation, which take place in the momentum space has no analogy in the coordinate space.

Some aspects of the anomalous diffusion in the velocity space have been investigated for the last decade in a few papers [14-18]. On the whole, comparing with the anomalous

diffusion in coordinate space, the anomalous diffusion in velocity space is weakly studied. The consequent way to describe the anomalous diffusion in the velocity space is, according to our knowledge, still absent.

In this paper the new kinetic equation for anomalous diffusion in velocity space is derived (see also [1]) on the basis of the appropriate expansion of PTF (in the spirit of the approach suggested in [2] for the diffusion in coordinate space) and some particular problems are investigated on this basis.

The diffusion in velocity space for the cases of normal and anomalous behavior of the PT function is presented in the Section II. Starting from the argumentation based on the Boltzmann type of the PTF, we derive the new kinetic equation, which in fact can be applied to the wide class of the PTF functions. The particular cases of anomalous diffusion for hard spheres collisions with the specific power-type prescribed distribution function of the light particles is analyzed in the Section III. The universal character of anomalous diffusion in velocity space is absent for this case. But for the general case of the power-type PTF with the different powers (which are not connected in advance) the universality takes a place. For this case the universal limitations for the existence of anomalous diffusion are found. In the Section IV the Boltzmann type equation is used to consider influence of the drift of the light particles on the PTF function and on the opportunity for anomalous transport of the heavy component.

II. DIFFUSION IN THE VELOCITY SPACE ON THE BASIS OF A MASTER-TYPE EQUATION

Let us consider now the main problem formulated in the introduction, namely, diffusion in velocity space (V-space) on the basis of the respective master equation, which describes the balance of grains coming in and out the point p at the moment t. The structure of this equation is formally similar to the master equation Eq. (**??**) in the coordinate space

$$\frac{df_g(\mathbf{p},t)}{dt} = \int d\mathbf{q} \left\{ W(\mathbf{q},\mathbf{p}+\mathbf{q}) f_g(\mathbf{p}+\mathbf{q},t) - W(\mathbf{q},\mathbf{p}) f_g(\mathbf{p},t) \right\}. \tag{1}$$

Of course, for coordinate space there is no conservation law, similar to that in the momentum space. The probability transition $W(\mathbf{p},\mathbf{p}')$ describes the probability for a grain with momentum $\mathbf{p}'$ (point $\mathbf{p}'$) to transfer from this point $\mathbf{p}'$ to the point $\mathbf{p}$ per unit time. The

momentum transferring is equal $\mathbf{q} = \mathbf{p}' - \mathbf{p}$. Assuming in the beginning that the characteristic displacements are small one may expand Eq. (**??**) and arrive at the Fokker-Planck form of the equation for the density distribution $f_g(\mathbf{p}, t)$

$$\frac{df_g(\mathbf{p}, t)}{dt} = \frac{\partial}{\partial p_\alpha}\left[A_\alpha(\mathbf{p}) f_g(\mathbf{p}, t) + \frac{\partial}{\partial p_\beta}\left(B_{\alpha\beta}(\mathbf{p}) f_g(\mathbf{p}, t)\right)\right]. \tag{2}$$

$$A_\alpha(\mathbf{p}) = \int d^s q q_\alpha W(\mathbf{q}, \mathbf{p}); \quad B_{\alpha\beta}(\mathbf{p}) = \frac{1}{2}\int d^s q q_\alpha q_\beta W(\mathbf{q}, \mathbf{p}). \tag{3}$$

The coefficients A_α and $B_{\alpha\beta}$ describing the friction force and diffusion, respectively.

Because the velocity of heavy particles is small, the $\mathbf{p}$-dependence of the PTF can be neglected for calculation of the diffusion, which in this case is constant $B_{\alpha\beta} = \delta_{\alpha\beta} B$, where B is the integral

$$B = \frac{1}{2s}\int d^s q q^2 W(q). \tag{4}$$

If to neglect the $\mathbf{p}$-dependence of the PTF at all we arrive to the coefficient $A_\alpha = 0$ (while the diffusion coefficient is constant). This neglecting, as well known is wrong, and the coefficient A_α for the Fokker-Planck equation can be determined by use the argument that the stationary distribution function is Maxwellian. On this way we arrive to the standard form of the coefficient $MTA_\alpha(p) = p_\alpha B$, which is one of the forms of Einstein relation. For the systems far from equilibrium this argument is not acceptable.

To find the coefficients in the kinetic equation, which are applicable also to slowly decreasing PT functions, let us use a more general way, based on the difference of the velocities of the light and heavy particles. For calculation of the function A_α we have take into account that the function $W(\mathbf{q}, \mathbf{p})$ is scalar and depends on $q, \mathbf{q}\cdot\mathbf{p}, p$. Expanding $W(\mathbf{q}, \mathbf{p})$ on $\mathbf{q}\cdot\mathbf{p}$ one arrive to the approximate representation of the functions $W(\mathbf{q}, \mathbf{p})$ and $W(\mathbf{q}, \mathbf{p}+\mathbf{q})$:

$$W(\mathbf{q}, \mathbf{p}) \simeq W(q) + \tilde{W}'(q)(\mathbf{q}\cdot\mathbf{p}) + \frac{1}{2}\tilde{W}''(q)(\mathbf{q}\cdot\mathbf{p})^2. \tag{5}$$

$$W(\mathbf{q}, \mathbf{p}+\mathbf{q}) \simeq W(q) + \tilde{W}'(q)\,(\mathbf{q}\cdot\mathbf{p}) + \frac{1}{2}\tilde{W}''(q)(\mathbf{q}\cdot\mathbf{p})^2 + q^2\tilde{W}'(q), \tag{6}$$

where $\tilde{W}'(q) \equiv \partial W(q, \mathbf{q}\cdot\mathbf{p})/\partial(\mathbf{qp})\mid_{\mathbf{q}\cdot\mathbf{p}=0}$ and $\tilde{W}''(q) \equiv \partial^2 W(q, \mathbf{q}\cdot\mathbf{p})/\partial(\mathbf{qp})^2\mid_{\mathbf{q}\cdot\mathbf{p}=0}$.

Then, with the necessary accuracy, A_α equals

$$A_\alpha(\mathbf{p}) = \int d^s q q_\alpha q_\beta p_\beta \tilde{W}'(q) = p_\alpha \int d^s q q_\alpha q_\alpha \tilde{W}'(q) = \frac{p_\alpha}{s}\int d^s q q^2 \tilde{W}'(q) \tag{7}$$

If for the function $W(\mathbf{q}, \mathbf{p})$ the equality $\tilde{W}'(q) = W(q)/2MT$ is fulfilled, then we arrive to the usual Einstein relation

$$MTA_\alpha(\mathbf{p}) = p_\alpha B \tag{8}$$

Let us check this relation for the Boltzmann collisions, which are described by the PT-function $W(\mathbf{q},\mathbf{p}) = w_B(\mathbf{q},\mathbf{p})$ [12]:

$$w_B(\mathbf{q},\mathbf{p}) = \frac{2\pi}{\mu^2 q}\int_{q/2\mu}^{\infty} du\, u\, \frac{d\sigma}{do}\left[\arccos\left(1-\frac{q^2}{2\mu^2u^2}\right), u\right] f_b(u^2+v^2-\mathbf{q}\cdot\mathbf{v}/\mu), \tag{9}$$

where ($\mathbf{p} = M\mathbf{v}$) and $d\sigma(\chi,u)/do$, μ and f_b are respectively the differential cross-section for scattering, the mass and distribution function for the light particles. In Eq. (9) we took into account the approximate equalities for the scattering of the light and heavy particles $q^2 \equiv (\triangle\mathbf{p})^2 = p'^2(1-cos\theta)$ and $\theta \simeq \chi$, where $p' = \mu u$ is the momentum of the light particle before collision.

For the equilibrium Maxvellian distribution f_b^0 the equality $\tilde{W}'(q) = W(q)/2MT$ is evident and we arrive to the usual Fokker-Planck equation in velocity space with the constant diffusion $D \equiv B/M^2$ and friction $\beta \equiv B/MT = DM/T$ coefficients, which satisfy the Einstein relation.

For some non-equilibrium situations the PTF can possess a long tail. In this case we have derive a generalization of the Fokker-Planck equation in spirit of the above consideration for the coordinate case, because the diffusion and friction coefficients in the form Eqs. (4),(7) diverge for large q if the functions have the asymptotic behavior $W(q) \sim 1/q^\alpha$ with $\alpha \le s+2$ and (or) $\tilde{W}'(q) \sim 1/q^\beta$ with $\beta \le s+2$.

Let us insert in Eq. (1) the expansions for W (as an example we choose $s = 3$, the arbitrary s can be considered by the similar way). With necessary accuracy we find

$$\frac{df_g(\mathbf{p},t)}{dt} = \int d\mathbf{q}\{f_g(\mathbf{p}+\mathbf{q},t)[W(q)+\tilde{W}'(q)\,(\mathbf{q}\cdot\mathbf{p})+ \\ \frac{1}{2}\tilde{W}''(q)(\mathbf{q}\cdot\mathbf{p})^2+q^2\tilde{W}'(q)] - f_g(\mathbf{p},t)[W(q)+\tilde{W}'(q)\,(\mathbf{q}\cdot\mathbf{p})+\frac{1}{2}\tilde{W}''(q)(\mathbf{q}\cdot\mathbf{p})^2]\} \tag{10}$$

After the Fourier-transformation $f_g(\mathbf{r}) = \int \frac{d\mathbf{p}}{(2\pi)^3} exp(i\mathbf{pr})f_g(\mathbf{p},t)$ Eq. (10) reads:

$$\frac{df_g(\mathbf{r},t)}{dt} = \int d\mathbf{q}\{exp(-i(\mathbf{qr})[W(q)-i\tilde{W}'(q)\,(\mathbf{q}\cdot\frac{\partial}{\partial\mathbf{r}}) \\ -\frac{1}{2}\tilde{W}''(q)(\mathbf{q}\cdot\frac{\partial}{\partial\mathbf{r}})^2] - [W(q)-i\tilde{W}'(q)\,(\mathbf{q}\cdot\frac{\partial}{\partial\mathbf{r}}) - \frac{1}{2}\tilde{W}''(q)(\mathbf{q}\cdot\frac{\partial}{\partial\mathbf{r}})^2]\}f_g(\mathbf{r},t) \tag{11}$$

We can rewrite this equation as [1]:

$$\frac{df_g(\mathbf{r},t)}{dt} = A(r)f_g(\mathbf{r}) + B_\alpha(r)\frac{\partial f_g(\mathbf{r},t)}{\partial\mathbf{r}_\alpha} + C_{\alpha\beta}(r)\frac{\partial^2 f_g(\mathbf{r},t)}{\partial\mathbf{r}_\alpha\partial\mathbf{r}_\beta} \tag{12}$$

where

$$A(r) = \int d\mathbf{q}[exp(-i(\mathbf{qr})) - 1]W(q) = 4\pi \int_0^\infty dqq^2 \left[\frac{sin\,(qr)}{qr} - 1\right] W(q) \quad (13)$$

$$B_\alpha \equiv r_\alpha B(r);\ B(r) = -\frac{i}{r^2} \int d\mathbf{q}\mathbf{q}\mathbf{r}[exp(-i(\mathbf{qr}) - 1]\tilde{W}'(q) =$$
$$\frac{4\pi}{r^2} \int_0^\infty dqq^2 \left[cos\,(qr) - \frac{sin(qr)}{qr}\right] \tilde{W}'(q) \quad (14)$$

$$C_{\alpha\beta}(r) \equiv r_\alpha r_\beta C(r) = -\frac{1}{2} \int d\mathbf{q} q_\alpha q_\beta [exp(-i(\mathbf{qr}) - 1]\tilde{W}''(q) \quad (15)$$

$$C(r) = -\frac{1}{2r^4} \int d\mathbf{q}(\mathbf{qr})^{\mathbf{2}}[exp(-i(\mathbf{qr}) - 1]\tilde{W}''(q) =$$
$$\frac{2\pi}{r^2} \int_0^\infty dqq^4 \left[\frac{2sin(qr)}{q^3 r^3} - \frac{2cos\,(qr)}{q^2 r^2} - \frac{sin(qr)}{qr} + \frac{1}{3}\right] \tilde{W}''(q) \quad (16)$$

For the isotropic function $f(\mathbf{r}) = f(r)$ one can rewrite Eq. (12) in the form

$$\frac{df_g(r,t)}{dt} = A(r)f_g(r) + B(r)r\frac{\partial f_g(r,t)}{\partial r} + C(r)r^2\frac{\partial^2 f_g(r,t)}{\partial r^2} \quad (17)$$

For the case of strongly decreasing PDF the exponent under the integrals for the functions $A(r)$, $B(r)$ and $C(r)$ can be expanded:

$$A(r) \simeq= -\frac{r^2}{6} \int d\mathbf{q}\, q^2 W(q);\ B(r) \simeq -\frac{1}{3} \int d\mathbf{q}\, q^2 \tilde{W}'(q);\ C(r) \simeq 0. \quad (18)$$

Practically the approximation $C(r) \simeq 0$ is always applicable (see below Sec.III) and the general kinetic equation (12) for the Fourier-transform of the velocity distribution function takes the form

$$\frac{df_g(\mathbf{r},t)}{dt} = A(r)f_g(\mathbf{r}) + B_\alpha(r)\frac{\partial f_g(\mathbf{r},t)}{\partial \mathbf{r}_\alpha}. \quad (19)$$

Then the simplified kinetic equation for the case of short-range on q-variable PTF (non-equilibrium, in general case) reads

$$\frac{df_g(r,t)}{dt} = A_0 r^2 f_g(r) + B_0 r\frac{\partial f_g(r)}{\partial r}, \quad (20)$$

where $A_0 \equiv -1/6 \int d\mathbf{q}\, q^2 W(q)$ and $B_0 \equiv -1/3 \int d\mathbf{q}\, q^2 \tilde{W}'(q)$.

The stationary solution of Eq. (17) for $C(r) = 0$ reads

$$f_g(r,t) = Cexp\left[-\int_0^r dr'\frac{A(r')}{r'B(r')}\right] = Cexp\left[-\frac{A_0 r^2}{2B_0}\right] \quad (21)$$

The respective normalized stationary momentum distribution equals

$$f_g(p) = \frac{N_g B_0^{3/2}}{(2\pi A_0)^{3/2}} exp\,[-\frac{B_0 p^2}{2A_0}] \tag{22}$$

Therefore in Eq. (21) the constant $C = N_g$. Equation (20) and this distribution are the generalization of the Fokker-Planck case for normal diffusion on non-equilibrium situation, when the prescribed $W(\mathbf{q}, \mathbf{p})$ is determined, e.g., by some non-Maxwellian distribution of the small particles f_b. To show this by other way let us make the Fourier transformation of (12) with $C = 0$ and the respective A and B_α:

$$\frac{df_g(\mathbf{p}, t)}{dt} = -A_0 \frac{\partial^2 f_g(\mathbf{p}, t)}{\partial p^2} - B_0 \frac{\partial (p_\alpha f_g(\mathbf{p}, t))}{\partial p_\alpha}, \tag{23}$$

Therefore we arrive to the Fokker-Planck type equation with the friction coefficient $\beta \equiv -B_0$ and diffusion coefficient $D = -A_0/M^2$. In general these coefficients (Eq. (18)) do not satisfy to the Einstein relation.

In the case of equilibrium W-function (e.g., $f_b = f_b^0$, see above) the equality $\tilde{W}'(q) = W(q)/2MT_b$ is fulfilled. Then with necessary accuracy (the second term in Eq. (18) with W' is of order μ/M and negligible in comparison with the first one) we find $A(r)/rB(r) \equiv A_0/B_0 = MT_b$. In this case the Einstein relation between the diffusion and friction coefficients $D = \beta T/M$ exists and the standard Fokker-Planck equation is valid.

III. THE MODELS OF ANOMALOUS DIFFUSION IN V - SPACE

Now we can calculate the coefficients for the models of anomalous diffusion.

At first we calculate the simple model system of the hard spheres with the different masses m and $M \gg m$, $d\sigma/do = a^2/4$. Let us suppose that in the model under consideration the small particles are described by the prescribed stationary distribution $f_b = n_b \phi_b / u_0^3$ (where ϕ_b is non-dimensional distribution, u_0 is the characteristic velocity for the distribution of the small particles) and $\xi \equiv (u^2 + v^2 - \mathbf{q}\cdot\mathbf{v}/\mu)/u_0^2$.

$$W_a(\mathbf{q}, \mathbf{p}) = \frac{n_b a^2 \pi}{2\mu^2 u_0 q} \int_{(q^2/4\mu^2 + v^2 - \mathbf{q}\cdot\mathbf{v}/\mu)/u_0^2}^{\infty} d\xi \,\cdot \phi_b(\xi). \tag{24}$$

If the distribution $\phi_b(\xi) = 1/\xi^\gamma$ ($\gamma > 1$) possess a long-tail we get

$$W_a(\mathbf{q}, \mathbf{p}) = \frac{n_b a^2 \pi}{2\mu^2 u_0 q} \frac{\xi^{1-\gamma}}{(1-\gamma)}|_{\xi_0}^{\infty} = \frac{n_b a^2 \pi}{2\mu^2 u_0 q} \frac{\xi_0^{1-\gamma}}{(\gamma - 1)}, \tag{25}$$

where $\xi_0 \equiv (q^2/4\mu^2 + v^2 - \mathbf{q}\cdot\mathbf{v}/\mu)/u_0^2$.

For the case $p = 0$ the value $\xi_0 \to \tilde{\xi}_0 \equiv q^2/4\mu^2 u_0^2$ and we arrive to the expression for anomalous $W \equiv W_a$

$$W_a(\mathbf{q}, \mathbf{p} = \mathbf{0}) = \frac{n_b a^2 \pi}{2^{3-2\gamma}(\gamma - 1)\mu^{4-2\gamma} u_0^{3-2\gamma} q^{2\gamma-1}} \equiv \frac{C_a}{q^{2\gamma-1}}. \tag{26}$$

To determine the structure of the transport process and the kinetic equation in the velocity space we have find also the functions $\tilde{W}'(q)$ and $\tilde{W}''(q)$.

If $p \neq 0$ to find $\tilde{W}'(q)$ and $\tilde{W}''(q)$ we have use the full value $\xi_0 \equiv (q^2/4\mu^2 + p^2/M^2 - \mathbf{q}\cdot\mathbf{p}/M\mu)/u_0^2$ and it derivatives on $\mathbf{q}\cdot\mathbf{p}$ at $p = 0$, $\xi_0' = -1/M\mu u_0^2$ and $\xi_0'' = 0$. Then

$$\tilde{W}'(\mathbf{q}, \mathbf{p}) \equiv \frac{n_b a^2 \pi}{2M\mu^3 u_0^3 q} \xi_0^{-\gamma}; \quad \tilde{W}''(\mathbf{q}, \mathbf{p}) \equiv \frac{n_b a^2 \pi \gamma}{2M^2 \mu^4 u_0^5 q} \xi_0^{-\gamma-1} \tag{27}$$

Therefore for $p = 0$ ($\xi_0 \to \tilde{\xi}_0$) we obtain the functions

$$\tilde{W}'(q) \equiv \frac{(4\mu^2 u_0^2)^\gamma n_b a^2 \pi}{2M\mu^3 u_0^3 q^{2\gamma+1}}; \quad \tilde{W}''(q) \equiv \frac{(4\mu^2 u_0^2)^{\gamma+1} n_b a^2 \pi \gamma}{2M^2 \mu^4 u_0^5 q^{2\gamma+3}} \tag{28}$$

The function $A(r)$ according to Eq. (13)

$$A(r) \equiv 4\pi \int_0^\infty dq q^2 \left[\frac{sin\,(qr)}{qr} - 1\right] W(q) = 4\pi C_a \int_0^\infty dq \frac{1}{q^{2\gamma-3}} \left[\frac{sin(qr)}{qr} - 1\right] \tag{29}$$

Comparing the reduced equation (see below) in the velocity space with the diffusion in coordinate space ($2\gamma - 1 \leftrightarrow \alpha$ and $W(q) = C/q^{2\gamma-1}$) we can establish that the convergence of the integral in the right side of Eq. (29) (3d case) is provided if $3 < 2\gamma - 1 < 5$ or $2 < \gamma < 3$. The inequality $\gamma < 3$ provides the convergence for small q ($q \to 0$) and the inequality $\gamma > 2$ provides the convergence for $q \to \infty$.

We have establish now the conditions of convergence the integrals for $B(r)$ and $C(r)$

$$B(r) = \frac{4\pi}{r^2} \int_0^\infty dq q^2 \left[cos\,(qr) - \frac{sin(qr)}{qr}\right] \tilde{W}'(q) \tag{30}$$

Convergence $B(r)$ exists for small q if $\gamma < 2$ and for large $q \to \infty$ for $\gamma > 1/2$.

Finally for $C(r)$ convergence is determined by the equalities $\gamma < 2$ for small q and $\gamma > 1$ for large q

$$C(r) = \frac{2\pi}{r^2} \int_0^\infty dq q^4 \left[\frac{2sin(qr)}{q^3 r^3} - \frac{2cos\,(qr)}{q^2 r^2} - \frac{sin(qr)}{qr} + \frac{1}{3}\right] \tilde{W}''(q) \tag{31}$$

Therefore to provide convergence for A, B, C for large q we have provide convergence for A, that means $\gamma > 2$. To provide convergence for small q it is enough to provide convergence

for B and C, that means $\gamma < 2$. Therefore for the purely power behavior of the function $f_b(\xi)$ convergence is absent. However, for existence of the anomalous diffusion in the momentum space in reality the convergence for small q is always provided, e.g., by finite value of v or by change of the small q-behavior of $W(q)$ (compare with the examples of anomalous diffusion in coordinate space [2]). Therefore, in the model under consideration, the "anomalous diffusion in velocity space" for the power behavior of $W(q)$, $\tilde{W}'(q)$ and $\tilde{W}''(q)$ on large q exists if for large q the asymptotic behavior of $W(q \to \infty) \sim 1/q^{2\gamma-1}$ with $\gamma > 2$. At the same time the expansion of the exponential function in Eqs. (13)-(16) under the integrals, which leads to the Fokker-Planck type kinetic equation is invalid for the power-type kernels $W(\mathbf{q}, \mathbf{p})$.

Let us consider now the formal general model for which we will not connect the functions $W(q)$, $\tilde{W}'(q)$ and $\tilde{W}''(q)$ with the concrete form of $W(\mathbf{q}, \mathbf{p})$. In this case one can suggest that the functions possess the independent one from another power-type q-dependence.

As an example, this dependence can be taken as the power type for three functions $W(q) \equiv a/q^{\alpha}$, $\tilde{W}'(q) \equiv b/q^{\beta}$ and $\tilde{W}''(q) \equiv c/q^{\eta}$, where α, β and η are independent and positive. Then as follows from the consideration above the convergence of the function W exists if $5 > \alpha > 3$ (for asymptotically small and large q respectively). For the function $\tilde{W}'(q)$ the convergence condition is $5 > \beta > 2$ for asymptotically small and large q respectively. Finally for the function $\tilde{W}''(q)$ the convergence condition is $7 > \eta > 5$ (for asymptotically small and large q respectively).

For this example the kinetic equation Eq. (12) reads

$$\frac{df_g(\mathbf{r},t)}{dt} = P_0 r^{\alpha-3} f(\mathbf{r},t) + r^{\beta-5} P_1 r_i \frac{\partial}{\partial r_i} f(\mathbf{r},t) + r^{\eta-7} P_2 r_i r_j \frac{\partial^2}{\partial r_i \partial r_j} f(\mathbf{r},t), \tag{32}$$

where

$$P_0 = 4\pi a \int_0^\infty d\zeta \zeta^{2-\alpha} \left[\frac{sin\,\zeta}{\zeta} - 1 \right] \tag{33}$$

$$P_1 = 4\pi b \int_0^\infty d\zeta q \zeta^{2-\beta} \left[cos\,\zeta - \frac{sin\zeta}{\zeta} \right] \tag{34}$$

$$P_2 = 4\pi c \int_0^\infty d\zeta \zeta^{4-\eta} \left[\frac{sin\zeta}{\zeta^3} - \frac{cos\,\zeta}{\zeta^2} - \frac{sin\zeta}{2\zeta} + \frac{1}{6} \right] \tag{35}$$

Taking into account the isotropy in r-space we can rewrite Eq. (32) in the form

$$\frac{df_g(r,t)}{dt} = P_0 r^{\alpha-3} f(r) + r^{\beta-4} P_1 \frac{\partial}{\partial r} f(r,t) + r^{\eta-5} P_2 \frac{\partial^2}{\partial r^2} f(r,t), \tag{36}$$

Naturally, Eqs. (32),(36) can be formally rewritten in momentum (or in velocity) space via the fractional derivatives of various orders. Therefore, as is easy to see, for the purely power behavior of the functions $W(q)$, $\tilde{W}'(q)$ and $\tilde{W}''(q)$ the solution with the convergent coefficients exists for the powers in the intervals mentioned above. The universal type of anomalous diffusion in velocity space in the case under consideration, therefore, exists if $5 > \alpha > 3$, $5 > \beta > 2$ and $7 > \eta > 5$. It appears even for the cases, when the functions $W(q)$, $\tilde{W}'(q)$ and $\tilde{W}''(q)$ have not the short-range cutting. Of cause the general description is also valid for the more complicated functions W, W' and W'' possessing the non-power short range parts.

Now let us take into account the important circumstance: usually in the problem under consideration there is a small parameter μ/M, which can simplify description of the velocity diffusion. As is easy to see, e.g., on the basis of the particular cases (e.g., Eq. (28)) for the convergent kernels of anomalous transport the term with second space derivative in general equations for distributions Eqs. (12),(17) is small in comparison with the term with the second derivative (as well as for the case of normal diffusion in velocity space). This smallness is of the order of the small ratio μ/M of the mass of the particles. Therefore for the most physically important kernels, describing the anomalous velocity diffusion the term with the second space derivative can be omitted and for non-stationary anisotropic and isotropic cases the diffusion equation respectively reads

$$\frac{df_g(\mathbf{r},t)}{dt} = A(r)f_g(\mathbf{r}) + B_\alpha(r)\frac{\partial f_g(\mathbf{r},t)}{\partial \mathbf{r}_\alpha} \tag{37}$$

and

$$\frac{df_g(r,t)}{dt} = A(r)f_g(r) + B(r)r\frac{\partial f_g(r,t)}{\partial r} \tag{38}$$

For the case of purely power behavior $W(q) = 1/q^\alpha$ and $W'(q) = 1/q^\beta$ we have, as above, $A(r) = P_0 r^{\alpha-3}$ and $rB(r) = P_1 r^{\beta-4}$ (with inequalities $5 > \alpha > 3$ and $5 > \beta > 2$). The stationary solution of Eq. (38)) (see, also (21)) for the case under consideration reads

$$f_g^{St}(r) = Cexp\,[-\int^r dr'\frac{A(r')}{r'B(r')}] = Cexp\,[-\frac{P_0 r^{\alpha-\beta+2}}{P_1(\alpha-\beta+2)}] \tag{39}$$

where $5 > \alpha - \beta + 2 > 0$.

To find the solution in isotropic non-stationary case Eq. (38) has to be written in the form

$$\frac{dX(r,t)}{dt} - B(r)r\frac{\partial}{\partial r}X(r,t) = A(r), \tag{40}$$

where $X(r,t) \equiv lnf_g(r,t)$. The general non-stationary solution of this equation can be written as the sum of general solution of the homogeneous equation $Y(r,t)$ (Eq. (40) in which $A(r)$ is taken equals zero)

$$Y(r,t) = \Phi(\xi),\ \xi \equiv t + \int_{r_0}^{r} dr' \frac{1}{r'B(r')}, \tag{41}$$

where Φ is the arbitrary function and the particular solution $Z(r,t)$ of the non-homogeneous equation Eq. (40):

$$Z(r,t) \equiv f_g^{St}(r) = -\int_{r_0}^{r} dr' \frac{A(r')}{r'B(r')} \tag{42}$$

Therefore

$$f_g(r,t) = exp\left[X(r,t)\right] \equiv exp\left[Y+Z\right] = L(\xi) f_g^{St}(r), \tag{43}$$

where $L(\xi)$ is the arbitrary function of ξ, which has to be found from the initial condition $f_g(r, t=0) \equiv \phi_0(r)$.

The variable $\xi(r,t)$ equals

$$\xi(r,t) = t + \int_{r_0}^{r} dr' \frac{1}{r'B(r')} = t + \frac{r^{5-\beta}}{P_1(5-\beta)} + c, \tag{44}$$

where c is the arbitrary constant, which can be omitted due to presence of the arbitrary function L and the values $5 > \alpha - \beta + 2 > 0$, $3 > 5 - \beta > 0$. The general non-stationary solution for the case under consideration reads

$$f_g(r,t) = L\left(t + \frac{r^{5-\beta}}{P_1(5-\beta)}\right) exp\left[-\frac{P_0 r^{\alpha-\beta+2}}{P_1(\alpha-\beta+2)}\right], \tag{45}$$

The unknown function L can be found from Eq. (45) and the initial condition $f_g(r,0) \equiv \phi_g(r)$:

$$L\left(\frac{r^{5-\beta}}{P_1(5-\beta)}\right) exp\left[-\frac{P_0 r^{\alpha-\beta+2}}{P_1(\alpha-\beta+2)}\right] = \phi_g(r), \tag{46}$$

The function $\phi_g(r) = \int d^3p exp(i\mathbf{pr}) f_g(p, t=0)$ is the Fourier-component of the initial distribution in momentum space. By use the notation $\zeta \equiv r^{5-\beta}/[P_1(5-\beta)]$ (what means $r(\zeta) \equiv [P_1(5-\beta)\zeta]^{1/(5-\beta)}$) we find

$$L(\zeta) = \phi_g[r(\zeta)]\, exp\left\{\frac{P_0 [r(\zeta)]^{\alpha-\beta+2}}{P_1(\alpha-\beta+2)}\right\}, \tag{47}$$

Therefore the time-dependent solution is equal

$$f_g(r,t) = \phi_g[r(\zeta+t)]\, exp\left\{\frac{P_0[r(\zeta+t)]^{\alpha-\beta+2}}{P_1(\alpha-\beta+2)}\right\} exp\left[-\frac{P_0 r^{\alpha-\beta+2}}{P_1(\alpha-\beta+2)}\right], \tag{48}$$

where we have express $r(\zeta+t) \equiv [P_1(5-\beta)(\zeta+t)]^{1/(5-\beta)}$ as a function of r, t:

$$r(\zeta+t) \equiv [P_1(5-\beta)(\zeta+t)]^{1/(5-\beta)} \equiv [r^{5-\beta} + P_1(5-\beta)t]^{1/(5-\beta)}, \tag{49}$$

or finally

$$f_g(r,t) = \phi_g\left([r^{5-\beta} + P_1(5-\beta)\,t]^{1/(5-\beta)}\right) exp\left\{\frac{P_0\,[r^{5-\beta} + P_1\,(5-\beta)\,t]^{(\alpha-\beta+2)/(5-\beta)} - P_0\,r^{\alpha-\beta+2}}{P_1\,(\alpha-\beta+2)}\right\} \tag{50}$$

It is necessary to stress that for the fractional powers $1/(5-\beta)$ and (or) $(\alpha-\beta+2)/(5-\beta)$ the real solution exists only if $P_1 > 0$. The limit $t \to \infty$ for the solution only for the specific initial conditions can coincide with the stationary solution.

For the power dependence of the functions $W(q)$ and $W'(q)$ the equation (37) can be formally written in fractional derivatives:

$$\frac{df_g(\mathbf{p},t)}{dt} = P_0 D^{\nu} f_g(\mathbf{p},t) - P_1(3+\gamma)D^{\gamma} f_g(\mathbf{p},t) + P_1 p_\alpha D_\alpha^{\gamma+1} f_g(\mathbf{p},t), \tag{51}$$

where $\nu \equiv \alpha - 3$, $\gamma \equiv \beta - 5$ and $D_\alpha^{\gamma+1} f_g(\mathbf{p},t) \equiv i \int d^3r exp(-i\mathbf{pr}))r_\alpha r^\gamma f_g(\mathbf{r},t)$.

The specific case of anomalous diffusion in velocity space, which leads to the equation similar to one in [14,15] is derived in the Appendix on the basis of general equation (12).

IV. THE MODEL OF DIFFUSION ON BASIS OF THE BOLTZMANN COLLISIONS WITH DRIFT DISTRIBUTION

Let us consider the simplest case of non-equilibrium, but stationary distribution f_b, namely the shifted velocity distribution.

The evident generalization of the PT-function for the case of the shifted velocity distribution of the light particles (with the drift velocity $\mathbf{u_d}$) the PT-function $w_B^d(\mathbf{q},\mathbf{p})$ ($\mathbf{p} = M\mathbf{v}$) reads

$$w_B^d(\mathbf{q},\mathbf{p},\mathbf{u_d}) = \frac{2\pi}{\mu^2 q}\int_{q/2\mu}^{\infty} duu \cdot \frac{d\sigma}{do}\left[\arccos\left(1-\frac{q^2}{2\mu^2u^2}\right), u\right] \times \\ f_b(u^2 + (\mathbf{v}-\mathbf{u_d})^2 - \mathbf{q}\cdot(\mathbf{v}-\mathbf{u_d})/\mu). \tag{52}$$

Again as in the Section II to find the coefficients in the kinetic equation let us use the way, based on the difference between the velocities of the light and heavy particles. At the same time the driven velocity u_d is not, generally speaking, small in comparison with the current characteristic velocities u and q/μ of the small particles.

For calculation of the function A_α we have take into account that in general the scalar function $w_B^d(\mathbf{q},\mathbf{p},\mathbf{u_d})$ has to be taken in the form $W(\mathbf{q},\mathbf{p},\mathbf{u_d}) = W(q,p,u_d,l,\xi,\eta)$ (here $l \equiv (\mathbf{q}\cdot\mathbf{p_d}))$ and expanded on $\xi \equiv (\mathbf{q}\cdot\mathbf{p})$ and $\eta \equiv (M\mathbf{u_d}\cdot\mathbf{p}) \equiv (\mathbf{p_d}\cdot\mathbf{p})$. In fact it is the expansion on velocity $\mathbf{v}$, which is small in comparison with other characteristic velocities q/μ, u and u_d. As we showed above (for the case $u_d = 0$) to arrive to the simple and solvable equation for the distribution f_g, taking into account smallness of v in comparison with the characteristic velocities and v^2 in comparison with $(\mathbf{q}\cdot\mathbf{v}/\mu))$ we have approximate the function $W(q,p,u_d,l,\xi,\eta) \simeq W(q,u_d,l,\xi,\eta)$, because we are interested mainly the high value of q for anomalous transport. At the same time after this type of neglecting and expansion on ξ and (for $u_d \neq 0$) η we can arrive for the special case of the kernels (e.g., purely power-type kernels on q, which are often considering for diffusion in coordinate space) to the divergence in some coefficients of the diffusion equation created by the region of a small q. This divergence is really absent for the realistic PT functions, which have cutting at small q. This cutting for small q has the physical reasons and is not related with the neglecting $v = p/M$ in the approximation $W(\mathbf{q},\mathbf{p},\mathbf{u_d}) \equiv W(q,p,u_d,l,\xi,\eta) \simeq W(q,u_d,l,\xi,\eta)$.

Let us expand $W(q,p,u_d,l,\xi,\eta)$:

$$W(\mathbf{q},\mathbf{p},\mathbf{u_d}) = W(q,p,u_d,l,\xi,\eta) \simeq W_0(q,p,u_d,l) + \partial W/\partial\xi \mid_{\xi,\eta=0} \xi + \partial W/\partial\eta \mid_{\xi,\eta=0} \eta + \frac{1}{2}\partial^2 W/\partial\xi^2 \mid_{\xi,\eta=0} \xi^2 + \frac{1}{2}\partial^2 W/\partial\eta^2 \mid_{\xi,\eta=0} \eta^2 + \partial^2 W/\partial\xi\partial\eta \mid_{\xi,\eta=0} \xi\eta, \quad (53)$$

where $W_0(q,p,u_d,l) \equiv W(q,p,u_d,l,\xi,\eta) \mid_{\xi,\eta=0}$. Then introducing $V_1(q,p,u_d,l) = \partial W/\partial\xi \mid_{\xi,\eta=0}, U_1(q,p,u_d,l) = \partial W/\partial\eta \mid_{\xi,\eta=0}, V_2(q,p,u_d,l) = \frac{1}{2}\partial^2 W/\partial\xi^2 \mid_{\xi,\eta=0}, U_2(q,p,u_d,l) = \frac{1}{2}\partial^2 W/\partial\eta^2 \mid_{\xi,\eta=0}, W_2(q,p,u_d,l) = \partial^2 W/\partial\xi\partial\eta \mid_{\xi,\eta=0}$ we can rewrite Eq. (53) in the form

$$W(q,p,u_d,l,\xi,\eta) \simeq W_0(q,p,u_d,l) + V_1(\mathbf{q}\cdot\mathbf{p}) + U_1(\mathbf{p_d}\cdot\mathbf{p}) + V_2(\mathbf{q}\cdot\mathbf{p})^2 + U_2(\mathbf{p_d}\cdot\mathbf{p})^2 + W_2(\mathbf{q}\cdot\mathbf{p})(\mathbf{p_d}\cdot\mathbf{p}) \simeq W_0(q,p=0,u_d,l) + V_1(q,p=0,u_d,l)(\mathbf{q}\cdot\mathbf{p}) + U_1(q,p=0,u_d,l)(\mathbf{p_d}\cdot\mathbf{p}), \quad (54)$$

Finally we put $p = 0$ in the coefficients of Eq. (54) and omit the terms with second order derivatives due to existence of the small parameter μ/M. It means only the terms of order

$\sqrt{\mu/M}$ are essential in the expansion of $W(\mathbf{q},\mathbf{p},\mathbf{u_d})$. Let us calculate $W(\mathbf{q},\mathbf{p}+\mathbf{q},\mathbf{u_d})$ taking into account the difference between the values of the characteristic momenta $\mathbf{q},\mathbf{p}+\mathbf{q},\mathbf{u_d}$ or velocities.

Then the expansion for the function $W(\mathbf{q},\mathbf{p}+\mathbf{q},\mathbf{u_d})$ with the necessary accuracy reads

$$W(\mathbf{q},\mathbf{p}+\mathbf{q},\mathbf{u_d}) \simeq W(\mathbf{q},\mathbf{p},\mathbf{u_d}) + \{q_\alpha \partial/\partial p_\alpha + \frac{1}{2} q_\alpha q_\beta \frac{\partial^2}{\partial p_\alpha \partial p_\beta}\} W(\mathbf{q},\mathbf{p},\mathbf{u_d}) \simeq \tag{55}$$

$$W_0(q,p=0,u_d,l) + V_1(q,p=0,u_d,l)(\mathbf{q}\cdot\mathbf{p}) + U_1(q,p=0,u_d,l)(\mathbf{p_d}\cdot\mathbf{p}) + V_1(q,p=0,u_d,l)\mathbf{q}^2 + U_1(q,p=0,u_d,l)(\mathbf{q}\cdot\mathbf{p_d}) \tag{56}$$

or

$$W(\mathbf{q},\mathbf{p}+\mathbf{q},\mathbf{u_d}) \simeq W_0(q,p=0,u_d,l) + V_1(q,p=0,u_d,l)[\mathbf{q}\cdot(\mathbf{p}+\mathbf{q})] + U_1(q,p=0,u_d,l)[\mathbf{p_d}\cdot(\mathbf{p}+\mathbf{q})]. \tag{57}$$

Then the kinetic equation reads

$$\frac{df_g(\mathbf{p},t)}{dt} = \int d\mathbf{q}\{[W_0(q,p=0,u_d,l) + V_1(q,p=0,u_d,l)(\mathbf{q}\cdot\mathbf{p}) + U_1(q,p=0,u_d,l)(\mathbf{p_d}\cdot\mathbf{p}) + V_1(q,p=0,u_d,l)\mathbf{q}^2 + U_1(q,p=0,u_d,l)(\mathbf{q}\cdot\mathbf{p_d})] f_g(\mathbf{p}+\mathbf{q},t) - [W_0(q,p=0,u_d,l) + V_1(q,p=0,u_d,l)(\mathbf{q}\cdot\mathbf{p}) + U_1(q,p=0,u_d,l)(\mathbf{p_d}\cdot\mathbf{p})] f_g(\mathbf{p},t)\} \tag{58}$$

After the Fourier-transformation $f_g(\mathbf{r}) = \int \frac{d\mathbf{p}}{(2\pi)^3} exp(i\mathbf{pr}) f_g(\mathbf{p},t)$ Eq. (10) reads:

$$\frac{df_g(\mathbf{r},t)}{dt} = \int d\mathbf{q}\{exp(-i\mathbf{qr})[W_0(q,p=0,u_d,l) - iV_1(q,p=0,u_d,l)(\mathbf{q}\cdot\frac{\partial}{\partial\mathbf{r}}) - iU_1(q,p=0,u_d,l)(\mathbf{p_d}\cdot\frac{\partial}{\partial\mathbf{r}})] f_g(\mathbf{r},t)) - [W_0(q,p=0,u_d,l) - iV_1(q,p=0,u_d,l)(\mathbf{q}\cdot\frac{\partial}{\partial\mathbf{r}}) - iU_1(q,p=0,u_d,l)(\mathbf{p_d}\cdot\frac{\partial}{\partial\mathbf{r}})] f_g(\mathbf{r},t))\} \tag{59}$$

Therefore

$$\frac{df_g(\mathbf{r},t)}{dt} = A_d(\mathbf{r}) f_g(\mathbf{r},t) + \mathbf{B}_d(\mathbf{r})\frac{\partial}{\partial\mathbf{r}} f_g(\mathbf{r},t), \tag{60}$$

where

$$A_d(\mathbf{r},\mathbf{p_d}) = \int d\mathbf{q}[exp(-i\mathbf{qr}) - 1] W_0(q,p=0,u_d,l), \tag{61}$$

$$\mathbf{B}_d(\mathbf{r},\mathbf{p_d}) = -i\int d\mathbf{q}[exp(-i\mathbf{qr})-1]\{V_1(q,p=0,u_d,l)\mathbf{q}+U_1(q,p=0,u_d,l)\mathbf{p}_d\} \equiv$$
$$\mathbf{r}B'_d(\mathbf{r},\mathbf{p_d})+\mathbf{p_d}B''_d(\mathbf{r},\mathbf{p_d}), \quad (62)$$

In more detail we consider this equation in the separate paper.

V. APPENDIX. ANOMALOUS VELOCITY DIFFUSION FOR THE SPECIFIC CASE B(R)=CONST, C(R)=0

Let us consider now formally the specific particular case of anomalous diffusion, when the specific structure of the PTF $W(\mathbf{q},\mathbf{p})$ provides a rapid (let say, exponential) decrease of the functions $\tilde{W}'(q)$ and $\tilde{W}''(q)$. Therefore the exponential function under the integrals in the coefficients $B(r)$ and $C(r)$ can be expanded, that means $B(r) = B_0$ and $C(r) \simeq 0$ respectively. At the same time the function $W(q) \equiv a/q^{\alpha}$ has a purely power dependence on q.

Then the kinetic equation Eq. (12) reads

$$\frac{df_g(\mathbf{r},t)}{dt} = P_0 r^{\alpha-3} f_g(\mathbf{r},t) + B_0 r_i \frac{\partial}{\partial r_i} f_g(\mathbf{r},t), \quad (63)$$

or formally in the momentum space

$$\frac{df_g(\mathbf{p},t)}{dt} = P_0 D^{\nu} f_g(\mathbf{p},t) - B_0 \frac{\partial}{\partial p_i}[p_i f_g(\mathbf{p},t)], \quad (64)$$

where $\nu \equiv (\alpha-3)$ $(2>\nu>0)$ and we introduced the fractional differentiation operator $D^{\nu} f(\mathbf{p},t) \equiv \int d\mathbf{r} r^{\nu} exp(-i\mathbf{pr}) f(\mathbf{r},t)$ in the momentum space to compare this equation with the similar one in [14]. The stationary solution of Eq. (63) is equal

$$f_g(r) = C exp\,[-\frac{P_0 r^{\nu-1}}{B_0}] \quad (65)$$

$$f_g(p) = C\int d^3 r exp(-i\mathbf{pr}) exp\,[-\frac{P_0 r^{\nu-1}}{B_0}] \equiv \frac{4\pi C}{p}\int_0^{\infty} dr r sin(pr) exp\,[-\frac{P_0 r^{\nu-1}}{B_0}] \quad (66)$$

For the case $\nu = 1$ we find $f(\mathbf{p}) = n_g \delta(\mathbf{p})$, therefore $C = n_g exp\,[P_0/B_0]/(2\pi)^3$. The similar consideration has to be used for other types of anomalous diffusion in velocity space. The physically important applications based on the physical models for $W(\mathbf{q},\mathbf{p})$ function will be considered separately.

VI. CONCLUSIONS

In this paper the problem of anomalous diffusion in momentum (velocity) space is consequently considered. The new kinetic equation for anomalous diffusion in velocity space is derived without suggestion about existence of the equilibrium stationary distribution function. Namely for the strongly non-equilibrium situations the long tails in PT-functions can manifest themselves. The model of anomalous diffusion in velocity space is described on the basis of the respective expansion of the kernel in master equation. The conditions of the convergence for the coefficients of the kinetic equation are found for the particular cases. The wide variety of the anomalous processes in velocity space exists, because even in isotropic case the three different coefficients in the general diffusion equation are present (one of them is usually negligible). The example of the Boltzmann kernel with the prescribed distribution function for the light particles is studied, in particular for the hard spheres interaction. In general the Einstein relation for such situation is not applicable, because the stationary state can be far from equilibrium. For the normal diffusion the friction and diffusion coefficient are explicitly found for the non-equilibrium case. For equilibrium case the usual Fokker-Plank equation is reproduced as the particular case.

The non-stationary and in general non-equilibrium for $t \to 0$ solution is found for the definite initial conditions.

The kinetic equation for the heavy particles distribution in the case of the prescribed distribution function for the light particles, which possess a drift velocity is derived in the suggested general approach.

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